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A SHAPE FACTOR SCHEME FOR POINT SOURCE CONFIGURATIONS

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Abstract—Shape factors for steady heat conduction from single bodies and for various configurations of such bodies are often presented in heat transfer textbooks. These presentations are not always clear.

Using a single sphere in an infinite medium as an example, a scheme is developed to obtain the shape factor for the hemisphere, for two spheres, a sphere near an isothermal surface and near an adiabatic surface.

This scheme can also be applied to other bodies such as ellipsoids, circular discs, pieces of wire and ribbons.

The introduction of a "shape resistance", as the reciprocal of the shape factor, and of a "distance function", characterizing the influence of the mutual distance of the bodies, simplifies the presentation of the scheme.

The overall shape resistance of a configuration can be obtained additively from the shape resistance of a single body and from the distance function.

Shape resistance and distance function for various bodies are presented in diagrams.

NOMENCLATURE

A, area;
a, b, axis of ellipsoid;
D, distance;
$\mathscr{D}$, distance function;
h, planting depth, $D/2$;
k, thermal conductivity;
l, length;
n, length in normal direction;
$\mathscr{R}$, shape resistance;
r, radius;
$\mathscr{S}$, shape factor;
t, temperature;
ϑ, non-dimensional temperature ratio, $(t - t_2)/(t_1 - t_2)$;
ϕ, heat flow.

Indices
1, boundary one;
2, boundary two;
a, adiabatic plane;
B ellipsoid body
i, isothermal plane;
r, normalized with radius;
s, sphere;
I, referring to source;
II, referring to sink, source.

INTRODUCTION

IN RECENT years, shape factors for steady heat conduction have found their way into heat transfer textbooks. Such factors were first mentioned by Langmuir *et al.* [1]. They offer the advantage of transferring all difficulties encountered in calculating heat flows through odd geometries into the determination of one single factor. Moreover, a shape factor, determined once, for a certain geometry can be successfully used for all physical phenomena occurring in the same geometry and governed by the Laplace equation.

From Fourier's equation for steady heat conduction

$$\mathrm{d}\phi = -k\frac{\partial t}{\partial n}\mathrm{d}A \tag{1}$$

and with a non-dimensional temperature ratio

$$\vartheta = (t - t_2)/(t_1 - t_2) \tag{2}$$

t_1 and t_2 being the given temperatures of the boundaries, we obtain the equation

$$\phi = -k(t_1 - t_2)\int_A(\partial\vartheta/\partial n)\,\mathrm{d}A. \tag{3}$$

In this equation (3) three factors are discerned: a property k, a temperature difference $(t_1 - t_2)$ and an integral. This integral only depends on the shape of the body and is therefore named shape factor $\mathscr{S}$:

$$\mathscr{S} = \int_A(\partial\vartheta/\partial n)\,\mathrm{d}A. \tag{4}$$

In some cases, shape factors presented in heat transfer literature are ambiguous or even incorrect. On the other hand, in mathematical or electric field literature a number of solutions of Laplace's equation are given for odd geometry problems in a complex form together with other physical parameters, thus masking the shape factor.

It will be shown here that shape factors for spherical bodies and their various configurations, in many cases follow a simple scheme. From the shape factor of a single sphere in an infinitely extended medium, factors for the hemisphere in a semi-infinite medium can be derived, or for two spheres' arrangements, or for spheres buried in the ground with isothermal or adiabatic surface. For this purpose the introduction of a shape resistance and a distance function is convenient and renders vivid results.

Bodies originating from point sources follow the same scheme.

1. SINGLE SPHERE IN AN INFINITELY EXTENDED MEDIUM

From Laplace's equation the shape factor for two concentric isothermal spherical shells, of temperature t_1 for the inner and t_2 for the outer shell, can be calculated from equation (4) as

$$\mathscr{S} = 4\pi \Big/ \left(\frac{1}{r_1} - \frac{1}{r_2}\right). \qquad (5)$$

Extending the radius r_2 of the outer shell (i.e. the outer isotherm) into infinity, we obtain the shape factor of a single isothermal sphere in an infinitely extended medium as

$$\mathscr{S}_s = 4\pi r \qquad (6a)$$

or in non-dimensional form

$$\left(\frac{\mathscr{S}_s}{r}\right) = (\mathscr{S}_s)_r = 4\pi. \qquad (6b)$$

The reciprocal of this factor will be called "shape resistance" of a sphere, $\mathscr{R}_s$:

$$\mathscr{R}_s = 1/\mathscr{S}_s = 1/4\pi r \qquad (7a)$$

or in non-dimensional form

$$r\mathscr{R}_s = 1/(\mathscr{S}_s)_r = 1/4\pi \qquad (7b)$$

2. HEMISPHERE IN THE SURFACE OF A SEMI-INFINITELY EXTENDED MEDIUM

This case immediately follows from section 1 by dividing a single sphere arrangement into two halves. From one half of a sphere consequently only half the heat is conducted away giving a shape factor for the hemispherical arrangement

$$\mathscr{S}_{\frac{1}{2}s} = \tfrac{1}{2}\mathscr{S}_s = \tfrac{1}{2}\cdot 4\pi r^* \qquad (8a)$$

and a shape resistance

$$\mathscr{R}_{\frac{1}{2}s} = 2\cdot\mathscr{R}_s. \qquad (8b)$$

* Purposely the factor 4π is kept in all equations in order to indicate the spherical origin and to prevent confusion with cylindrical geometries where 2π is a characteristic factor.

It is important to note that the two isotherms t_1 and t_2 necessary for the solution of the heat conduction problem are formed, for one side, by the hemispherical shell with a given radius $r = r_1$, and for the other, by a hemispherical shell located at infinity $r \to \infty$. The plane dividing the original spheric arrangement, usually considered as the surface of the ground containing the hemisphere, originates from streamlines and therefore must be an adiabatic plane. As an isothermal surface, occasionally presented in literature, it would bring about the intersection of two isotherms and a thermal short circuit.

3. TWO SPHERES IN AN INFINITELY EXTENDED MEDIUM

From equations (3), (4) and (6a), with temperature t_2 being taken zero at infinity, the temperature field around a single sphere is characterised by

$$t = -\frac{\phi}{k}\frac{1}{4\pi r}. \qquad (9)$$

The temperature field around two spheres in a mutual distance $D = 2h$ is obtained by superposition as

$$t = t_{\mathrm{I}} + t_{\mathrm{II}} = -\frac{1}{4\pi}\cdot\frac{1}{k}\left(\frac{\phi_{\mathrm{I}}}{r_{\mathrm{I}}} + \frac{\phi_{\mathrm{II}}}{r_{\mathrm{II}}}\right) \qquad (10)$$

r_{I} and r_{II} being radius vectors.

Assuming a point source and a point sink of opposite equal capacity $\phi_{\mathrm{I}} = -\phi_{\mathrm{II}} = \phi$, the temperature anywhere in the field is

$$t = -\frac{\phi}{k}\frac{1}{4\pi}\left(\frac{1}{r_{\mathrm{II}}} - \frac{1}{r_{\mathrm{II}}}\right). \qquad (11)$$

Such a field giving heat flow lines as streamlines and isotherms as equipotential-lines is presented in Fig. 1.

For the case of opposite equal capacity, shown here, the isotherms flatten in the section where the two spheres face each other and they stretch in the averted parts, thus forming a pear-like pattern.

For the different case of two sources of equal capacity the isotherms flatten in the averted sections and stretch where the spheres face each other, until they finally merge. In some distance around the two sources then the isotherms will be ellipsoid-shaped.

From Fig. 1 it is observed that two spherical isotherms are obtained when the distance D is large compared to the radius r. For a distance–radius ratio $D/r \geqslant 5$ the surface temperatures t_1 and t_2 of the spheres can be calculated in good approximation by introducing for the radius vectors $r_2 \approx D$ and $r_1 \approx D$ respectively, giving

$$t_1 = -\frac{\phi}{k}\frac{1}{4\pi}\left(\frac{1}{r} - \frac{1}{D}\right) \text{ and } t_2 = -\frac{\phi}{k}\frac{1}{4\pi}\left(\frac{1}{D} - \frac{1}{r}\right). \qquad (12)$$

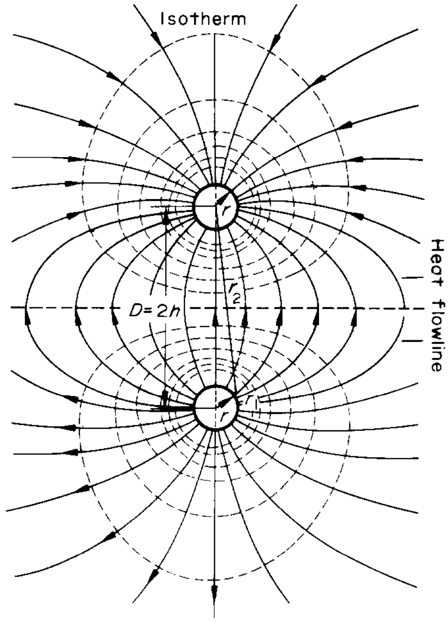


FIG. 1. Field lines around two spheres of opposite equal capacity.

The heat conducted between the two spheres is

$$\phi = -k(t_1 - t_2)\frac{4\pi r}{2(1 - r/D)} \tag{13}$$

and, by comparison with equations (3) and (4), shape factor and shape resistance are

$$\mathscr{S}_{2s} = \frac{4\pi r}{2(1 - r/D)};\ \mathscr{R}_{2s} = 2\cdot\frac{1}{4\pi r}(1 - r/D). \tag{14}$$

The influence of the distance D can be characterized separately by a so-called "distance-function"

$$\mathscr{D}_s = \frac{1}{4\pi}\cdot\frac{1}{D}. \tag{15}$$

Introducing this into the shape resistance of equation (14) we obtain

$$\mathscr{R}_{2s} = 2(\mathscr{R}_s - \mathscr{D}_s). \tag{16}$$

Thus the overall resistance $\mathscr{R}_{2s}$ may be considered as being composed of two partial resistances: the resistance $\mathscr{R}_s$ imposed by each single sphere and the distance function, which reduces this sphere resistance, to account for the fact that heat is conducted across a finite distance and not across an infinite distance for which $\mathscr{R}_s$ was derived. The factor 2 may indicate that two spheres are involved in our problem.

In literature, the case of small distance-radius ratios $2 < D/r < 5$ is also treated and in [2, 3] electrical potentials are calculated. From these, the shape factor is determined as the series

$$S_{2s} = \frac{4\pi r}{2}\left[1 + \frac{r}{D} + \left(\frac{r}{D}\right)^2 + \left(\frac{r}{D}\right)^3 + 2\left(\frac{r}{D}\right)^4 + 3\left(\frac{r}{D}\right)^5 + \ldots\right] \tag{17}$$

Comparison of shape factors calculated from equation (14) and (17) brings deviations of not more than about 1 per cent, even for small ratios of D/r ($\gtrsim 2$).

For very large values D/r or a very large distance D of the two spheres, the shape factor in (14) and (17) will approach the value of the hemispherical case. This appears reasonable imagining flow lines and their increasing lengths for the averted sections of the spheres.

4. SPHERES OPPOSITE TO A PLANE

In literature, these cases are described as spheres buried in the ground in some distance h to the surface.

With the sphere, in any case, forming an isotherm, two variations have to be distinguished: the plane surface of the ground representing an isotherm or an adiabate.

4.1 *Isothermal plane*

From Fig. 1 it is obvious that half way, $h = D/2$, between the source and the sink a straight, plane isotherm is located. Taking this plane isotherm as the surface of the ground with heat being conducted from the sphere to this surface, the length of all streamlines, of the two sphere case, now is cut in half. If now the temperature t_2 is associated to the plane, the heat flow will be twice as large as for two spheres and consequently the shape factor will be twice as large.

$$\mathscr{S}_{si} = 2\cdot\mathscr{S}_{2s} = 4\pi r/(1 - r/D). \tag{18}$$

The shape resistance for an isothermal sphere near an isothermal plane is

$$\mathscr{R}_{si} = \tfrac{1}{2}\cdot\mathscr{R}_{2s} = \mathscr{R}_s - \mathscr{D}_s. \tag{19}$$

In comparison to the two spheres' case with a resistance given in equation (16), the overall shape resistance in equation (19) is composed of the single difference of the shape factor of a single sphere and a distance function thus vividly demonstrating that only one sphere is considered now. The distance function has to be formed with $D = 2h$.

4.2 *Adiabatic plane*

Different from the source and sink arrangement used so far, now two sources of equal capacity $\phi_{\mathrm{I}} = \phi_{\mathrm{II}} = \phi$ shall be considered. From equation

(11) with the approximation $r_2 \approx D$ and $r_1 \approx D$ respectively the heat flow from either source to the infinite distant sink is calculated as

$$\phi = -kt\frac{4\pi r}{1 + r/D}. \quad (20)$$

Half way, $h = D/2$, between the two sources, again a straight plane is formed, which, originating from streamlines now, is an adiabatic surface.

Thus, the shape factor read from equation (20) applies to the isothermal sphere at a distance of $h = D/2$ of an adiabatic plane. This shape factor is

$$\mathcal{S}_{sa} = 4\pi r/(1 + r/D) \quad (21a)$$

and the shape resistance is

$$\mathcal{R}_{sa} = \frac{1}{4\pi r}(1 + r/D) = \mathcal{R}_s + \mathcal{D}_s. \quad (21b)$$

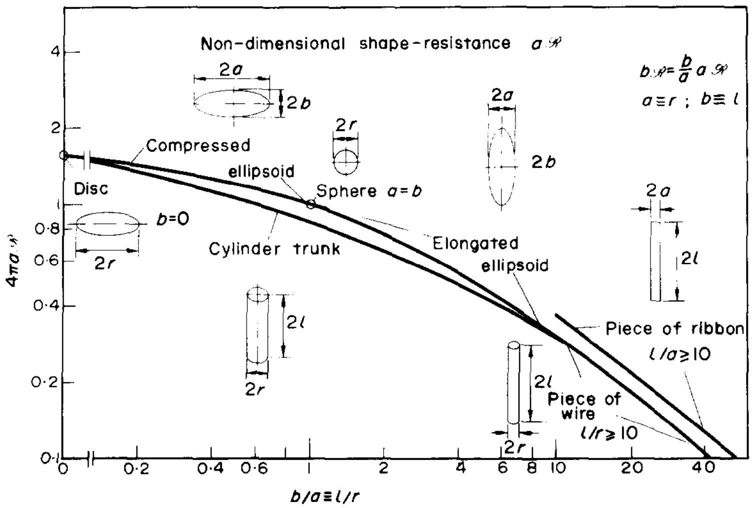


FIG. 2. Non-dimensional shape resistance for various bodies.

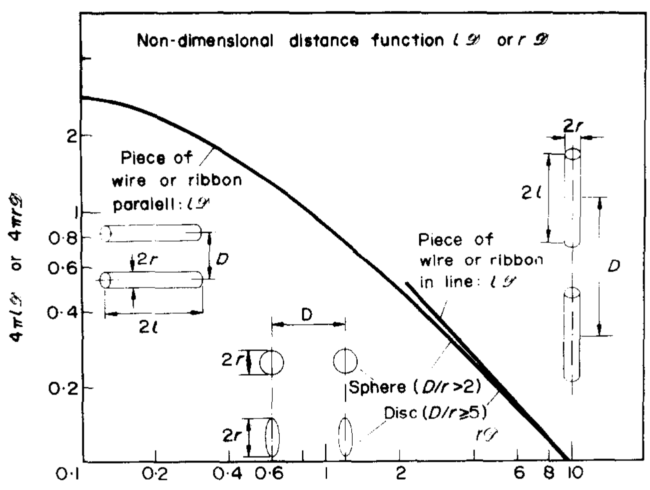


FIG. 3. Non-dimensional distance function for various arrangements of different bodies.

Comparing to equations (16) and (19), now the overall resistance is the sum of the single sphere resistance and the distance function. This addition of resistances may be explained with the compression of streamlines near the adiabatic boundary.

For large planting depths h, according to equation (15), the distance function becomes negligible and the resistance for both the isothermal and the adiabatic plane approach the value of the resistance of a single sphere.

5. BODIES ORIGINATING FROM POINT SOURCES

Point sources, lined up in a straight line will form a line source of finite length.

The isotherms around such a source have the shape of an elongated "cigar-like" ellipsoid. Long ellipsoids with small shorter axes ($b/a \gg 1$) can well be considered as pieces of a cylindric wire or as pieces of a flat ribbon.

Rotating an elongated ellipsoid around its shorter axis produces a compressed "discus-like" ellipsoid. With its shorter axis $b = 0$, this ellipsoid turns into a flat circular disc, and with axes $a = b$, a sphere is obtained.

Potential field solutions for such single bodies in an infinitely extended medium and respective two body arrangements are given in literature [2–8]. From those, a non-dimensional shape resistance and a non-dimensional distance function was determined and is presented in Figs. 2 and 3.

In Fig. 2 a non-dimensional shape resistance $a\mathcal{R}$ is plotted vs the ratio of the axes of the ellipsoid bodies, or for wires, vs the length–radius ratio.

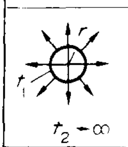
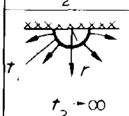
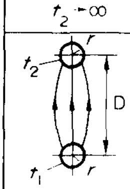
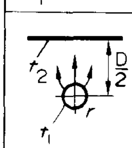
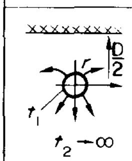
In order to read the shape resistance directly from Fig. 2, the ordinate values of $a\mathcal{R}$ are multiplied by 4π. This gives, e.g. for the flat disc with $b/a = 0$ and $a = r$

$$4\pi r\mathcal{R} = 1{\cdot}5708 = \frac{\pi}{2} \text{ and } \mathcal{R} = \frac{1}{8r}.$$

In Fig. 3, the non-dimensional distance function $l\mathcal{D}$ or $r\mathcal{D}$ is plotted versus a non-dimensional distance D/l or D/r. The normalization is based upon the greatest length occurring in the considered body. Ordinate and abscissa values have to correspond in their normalizing length, l or r.

From Figs. 2 and 3 the overall shape resistance for various bodies originating from point sources can be calculated following the scheme given for the example of spheres and compiled in Table 1.

Table 1.

Geometric arrangement		Shape factor	Shape resistance
	single body in infinite medium	$\mathcal{S}_B$ non-dimensional $(\mathcal{S}_B)_r = \frac{\mathcal{S}_B}{r}$	$\mathcal{R}_B$ non-dimensional $r\mathcal{R}_B = \frac{1}{(\mathcal{S}_B)_r}$
	semibody in semi-infinite medium	$\mathcal{S}_{\frac{1}{2}B} = \frac{1}{2}\mathcal{S}_B$	$\mathcal{R}_{\frac{1}{2}B} = 2\mathcal{R}_B$
	two bodies in infinite medium	$\mathcal{S}_{2B} = \frac{1}{2(\mathcal{R}_B - \mathcal{D}_B)}$ non-dimensional $(\mathcal{S}_{2B})_r = \frac{1}{2r(\mathcal{R}_B - \mathcal{D}_B)}$	$\mathcal{R}_{2B} = (2\mathcal{R}_B - \mathcal{D}_B)$ non-dimensional $r\mathcal{R}_{2B} = 2(r\mathcal{R}_B - r\mathcal{D}_B)$
	body near an isothermal plane	$\mathcal{S}_{Bi} = 2\mathcal{S}_{2B} = \frac{1}{\sqrt{\mathcal{R}_B - \mathcal{D}_B}}$	$\mathcal{R}_{Bi} = \frac{1}{2}\mathcal{R}_{2B} = \mathcal{R}_B - \mathcal{D}_B$
	body near an adiabatic plane	$\mathcal{S}_{Ba} = \frac{1}{\mathcal{R}_B + \mathcal{D}_B}$	$\mathcal{R}_{Ba} = \mathcal{R}_B + \mathcal{D}_B$

As indicated in Table 1, in the two body case, the normalizing lengths, have to be the same for either term. Alterations in these normalizing lengths can be performed with

$$l\mathscr{R} = \frac{l}{r}\cdot r\mathscr{R}. \tag{22}$$

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UNE MÉTHODE DE FACTEUR DE FORME POUR DES CONFIGURATIONS DE SOURCES PONCTUELLES

Résumé—Des facteurs de forme pour la conduction thermique dans des corps de forme variée sont présentés de plus en plus fréquemment dans les livres. Ces présentations ne sont pas toujours claires.

Prenant pour exemple une sphère unique dans un milieu infini on développe une méthode pour obtenir le facteur de forme pour l'hémisphère, pour deux sphères, pour une sphère près d'une surface isotherme et près d'une surface adiabatique.

Cette méthode peut être appliquée à d'autres corps tels que des ellipsoïdes, des disques circulaires, des tronçons de fils et de rubans.

L'introduction d'une "résistance de forme" comme inverse du facteur de forme et d'une "fonction de distance", simplifie la présentation de la méthode. La résistance globale de forme d'une configuration peut être obtenue de façon additive à partir de la résistance de forme d'un élément et de la fonction de distance.

On présente par des diagrammes la résistance de forme et la fonction de distance pour différents corps.

EIN FORMFAKTORSCHEMA FÜR PUNKTQUELLEN-ANORDNUNGEN

Zusammenfassung—Formfaktoren der stationären Wärmeleitung von Einzelkörpern und verschiedener ihrer Anordnungen werden immer häufiger in Lehrbüchern der Wärmeübertragung angegeben. Diese Angaben sind nicht immer eindeutig.

Am Beispiel einer Einzelkugel im unendlich ausgedehnten Medium wird ein Schema zur Ermittlung des Formfaktors angegeben, für die Halbkugel, für zwei Kugeln, einer Kugel vor einer isothermen und vor einer adiabaten Wand.

Dieses Schema lässt sich auch auf andere Körper, wie Ellipsoide, Kreisscheiben, Drahtstücke und Bänder anwenden.

Die Einf⁴hrung eines "Formwiderstandes" als dem Reziprokwert des Formfaktors und einer "Abstandsfunktion" als Charakteristkum des gegenseitigen Abstandes der Körper erleichert die Schematik. Der Gesamtformwiderstand einer Anordnung kann additiv aus dem Formwiderstand des Einzelkörpers und der Abstandsfunktion erhalten werden.

Formwiderstand und Abstandsfunktion sind für verschiedene Körper in Diagrammen wiedergegeben.

МЕТОД ФОРМ-ФАКТОРА ДЛЯ КОНФИГУРАЦИЙ ТОЧЕЧНЫХ ИСТОЧНИКОВ

Аннотация—В учебниках по теплообмену все чаще приводятся форм-факторы для стационарной теплопроводности одиночных тел различных конфигураций в форме, не всегда понятной читателю.

На примере одиночной сферы в бесконечной среде разрабатывается схема получения форм-фактора для полусферы, двух сфер, сферы вблизи изотермической поверхности и сферы вблизи адиабатической поверхности.

Этот метод также можно применить для таких тел, как эллипсоиды, круглые диски, куски проволоки и ленты.

Введение понятий сопротивления формы (обратной величины форм-фактора) и функции расстояния, характеризующей влияние расстояния между телами, значительно упрощает методику.

Суммарное значение сопротивления формы для системы любой конфигурации можно получить, используя значение сопротивления формы одиночного тела и функции расстояния.

Сопротивление формы и функция расстояния для разных тел представлены на графиках.